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Coupled intervals for discrete symplectic systems

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Abstract

In this paper we introduce (strict) coupled intervals for discrete symplectic systems and characterize in terms of the nonexistence of such coupled intervals the definiteness of the associated discrete quadratic functional with variable endpoints. This (strict) coupled interval notion generalizes (i) the (strict) conjugate interval notion known for discrete variational problems with fixed right endpoint, and (ii) the (strict) coupled interval notion known for the special case of linear Hamiltonian systems. The applicability of this theory of coupled intervals is clearly illustrated by a numerical example emanating from a nonlinear discrete control problem.

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1. Introduction and motivation

In recent years, several papers appeared which study the oscillatory properties of discrete symplectic systems of the form

$$z_{k+1} = S_k z_k, \quad k \in [0, N] \tag{S}$$

and definiteness of the corresponding quadratic functional

$$\mathscr{F}_0(z) := \sum_{k=0}^{N} z_k^{\mathrm{T}} \left\{ S_k^{\mathrm{T}} \mathscr{K} S_k - \mathscr{K} \right\} z_k.$$

The investigation of such systems was initiated in the monograph [1, Chapter 3] and pursued in [2,3,5–10,13–15,18]. The name *discrete symplectic system* is motivated by the requirement that the system matrix S_k is symplectic for all k. Examples of such systems are the discrete Hamiltonian systems (H) below (which already include higher order Sturm–Liouville difference equations and also second order matrix difference equations) and the discrete trigonometric or self-reciprocal systems, see e.g. [2,6] and Example 1.

The functional above arises as second variation in the discrete calculus of variations and control problems, so it is important to understand conditions characterizing its nonnegativity and positivity.

In the context of this paper, $z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}$ is a real $2n$-vector, $S_k = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix}$ is a real symplectic $2n \times 2n$ matrix, i.e., $S_k^{\mathrm{T}} \mathscr{J} S_k = \mathscr{J}$ with $\mathscr{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ being the $2n \times 2n$ skew-symmetric matrix, $\mathscr{K} := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ is a $2n \times 2n$ matrix (all blocks of $2n \times 2n$ matrices above have dimensions $n \times n$), and $[0, N]$ is a discrete interval with indicated endpoints.

There are several notions used in the characterizations of the nonnegativity and positivity of $\mathscr{F}_0$, namely the generalized zeros of vector solutions of (S) in [5], focal points of matrix solutions (conjoined bases) $Z_k = \begin{pmatrix} X_k \\ U_k \end{pmatrix}$ of (S) involving the so-called "kernel condition" $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ in [5,13–15], conjugate intervals in [13,15], and the so-called "image condition" $x_k \in \operatorname{Im} X_k$ in [8,9]. It was also shown in these papers that the oscillation theory of discrete Hamiltonian systems

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^{\mathrm{T}} u_k, \qquad k \in [0, N], \tag{H}$$

where B_k and C_k are symmetric and $I - A_k$ is invertible, extends naturally to discrete symplectic systems (S).

Recently, the authors derived in [16] new characterizations of the definiteness of the Hamiltonian quadratic functional associated with system (H) via the notion of a (strict) *coupled interval*. This notion replaces that of a conjugate interval, known for problems with fixed right endpoint, in problems with more general boundary conditions, that is, separable and jointly varying endpoints. However, for discrete symplectic systems parallel results are unknown.

The purpose of this paper is to extend the above mentioned Hamiltonian coupled interval notion to discrete symplectic system (S) and relate it to the definiteness of the corresponding quadratic functional. The main difficulty in this generalization lies in the fact that the evolution matrix $\Phi_{k+1,j} := \mathscr{A}_k \mathscr{A}_{k-1} \dots \mathscr{A}_j$ may be *singular* in the general symplectic case, as opposed to the Hamiltonian case where the matrices $\mathscr{A}_k = (I - A_k)^{-1}$ are invertible.

The paper is organized as follows. After introducing necessary notation and terminology, we define (strict) coupled intervals for the symplectic system (S). We characterize the definiteness of the corresponding quadratic functionals with *separated endpoints* via the nonexistence of such (strict) coupled intervals. Then we generalize this coupled interval notion to problems with *jointly varying endpoints* and obtain parallel results as for the separated endpoints case. To illustrate the applicability of the coupled intervals theory, an example is provided in Section 5 in which the positivity of the second variation is nicely verified via the nonexistence of coupled intervals.

Let us finally remark that this work is motivated by the continuous-time coupled points theory developed in [19,20], which hopefully will be unified with the present work in the framework of time scales, see e.g. [11,12].

2. Notation and terminology

It is convenient to write the system (S) in block entries, that is

$$x_{k+1} = \mathscr{A}_k x_k + \mathscr{B}_k u_k, \quad u_{k+1} = \mathscr{C}_k x_k + \mathscr{D}_k u_k. \tag{S}$$

The property that S_k (and S_k^{T}) is a symplectic matrix means that the coefficients satisfy

$$\begin{gathered} \mathscr{A}_k^{\mathrm{T}} \mathscr{D}_k - \mathscr{C}_k^{\mathrm{T}} \mathscr{B}_k = \mathscr{A}_k \mathscr{D}_k^{\mathrm{T}} - \mathscr{B}_k \mathscr{C}_k^{\mathrm{T}} = I, \\ \mathscr{A}_k \mathscr{B}_k^{\mathrm{T}},\ \mathscr{C}_k \mathscr{D}_k^{\mathrm{T}},\ \mathscr{C}_k^{\mathrm{T}} \mathscr{A}_k,\ \mathscr{D}_k^{\mathrm{T}} \mathscr{B}_k \text{ symmetric.} \end{gathered} \tag{1}$$

As any symplectic matrix is invertible, we have $S_k^{-1} = \begin{pmatrix} \mathscr{D}_k^{\mathrm{T}} & -\mathscr{B}_k^{\mathrm{T}} \\ -\mathscr{C}_k^{\mathrm{T}} & \mathscr{A}_k^{\mathrm{T}} \end{pmatrix}$ and the equivalent time-reversed system is

$$x_k = \mathscr{D}_k^{\mathrm{T}} x_{k+1} - \mathscr{B}_k^{\mathrm{T}} u_{k+1}, \quad u_k = -\mathscr{C}_k^{\mathrm{T}} x_{k+1} + \mathscr{A}_k^{\mathrm{T}} u_{k+1}.$$

Consequently, solutions of (S) are uniquely determined by their values at one index k. In this work we will freely identify the vector solutions $z = \begin{pmatrix} x \\ u \end{pmatrix}$ and the matrix solutions $Z = \begin{pmatrix} X \\ U \end{pmatrix}$ of (S) with the pairs (x, u) and (X, U), respectively.

A *conjoined basis* of (S) is a matrix solution (X, U) such that $X_k^{\mathrm{T}} U_k$ is symmetric and $\operatorname{rank} \begin{pmatrix} X_k \\ U_k \end{pmatrix} = n$ at some (and hence at any) index $k \in [0, N+1]$. The *principal solution* is the conjoined basis $(\widehat{X}, \widehat{U})$ of (S) starting with the initial values $\widehat{X}_0 = 0$ and $\widehat{U}_0 = I$. A conjoined basis (X, U) of (S) has *no focal points in* $(m, m+1]$ if

$$\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m, \quad P_m := X_m X_{m+1}^{\dagger} \mathscr{B}_m \geqslant 0, \tag{2}$$

where $\dagger$ stands for the Moore–Penrose generalized inverse of the given matrix. The notion of a *focal point* for a conjoined basis (X, U) is defined in the opposite way to (2). Namely, (X, U) has a focal point in $m+1$ if $\operatorname{Ker} X_{m+1} \nsubseteq \operatorname{Ker} X_m$, and in $(m, m+1)$ if $\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m$ but $P_m \ngeqslant 0$. In [18], Kratz generalized this focal point definition so as to include the *multiplicities* of focal points. The latter definition is, however, not needed in the present paper. For further details we refer e.g. to [5,15,18].

A pair (x, u) is *admissible* if $x_{k+1} = \mathscr{A}_k x_k + \mathscr{B}_k u_k$ for all $k \in [0, N]$. We will study the definiteness of quadratic functionals over such admissible pairs satisfying in addition certain *boundary conditions*. Namely, we will consider *separated* boundary conditions $\mathscr{M}_0 x_0 = 0$, $\mathscr{M}_1 x_{N+1} = 0$ with $n \times n$ projections $\mathscr{M}_0$, $\mathscr{M}_1$ and the associated (symmetric) $n \times n$ cost matrices Γ_0, Γ_1

satisfying $\Gamma_i = (I - \mathscr{M}_i)\Gamma_i(I - \mathscr{M}_i)$, $i = 0, 1$. In this context, the principal solution of (S) is replaced by the *natural conjoined basis* (X, U) of (S) which is given by the initial conditions $X_0 = I - \mathscr{M}_0$, $U_0 = \Gamma_0 + \mathscr{M}_0$. Note that $(X, U) = (\widehat{X}, \widehat{U})$ when the left endpoint is fixed, i.e., when $\mathscr{M}_0 = I$. Finally, we will deal with general *joint* boundary conditions $\mathscr{M}\begin{pmatrix} x_0 \\ x_{N+1}\end{pmatrix} = 0$, with $2n \times 2n$ projection $\mathscr{M}$, and the associated (symmetric) $2n \times 2n$ cost matrix Γ satisfying $\Gamma = (I - \mathscr{M})\Gamma(I - \mathscr{M})$.

An interval $(m, m + 1]$ is (*strictly*) *conjugate to* 0 if there exists a solution (x, u) of (S) satisfying the initial boundary and transversality conditions

$$\mathscr{M}_0 x_0 = 0, \quad u_0 = \Gamma_0 x_0 + \mathscr{M}_0\gamma_0 \tag{3}$$

for some vector $\gamma_0 \in \mathbb{R}^n$, and there is a vector $c \in \mathbb{R}^n$ such that

$$x_m \neq 0, \quad x_{m+1} = \mathscr{B}_m c, \quad d_m \leqslant 0 \; (d_m < 0),$$

where $d_m = d_m(x_m, c) := x_m^{\mathrm{T}} c$.

System (S) is $(\mathscr{M}_0 : I)$-*normal on* $[0, N + 1]$ if the only solution of $u_{k+1} = \mathscr{D}_k u_k$, $\mathscr{B}_k u_k = 0$, $k \in [0, N]$, satisfying $u_0 = \mathscr{M}_0\gamma_0$ for some vector $\gamma_0 \in \mathbb{R}^n$ is the zero sequence $u_k \equiv 0$ on $[0, N + 1]$.

Define the transition matrix $\Phi_{k,j}$ by $\Phi_{k,j} := \mathscr{A}_{k-1}\mathscr{A}_{k-2}\ldots\mathscr{A}_j$ for $k > j$ and $\Phi_{k,k} := I$.

In this paper, we will use the convention that $\sum_{k=a}^{b} := 0$ if $a > b$.

The following lemma is a straightforward calculation.

Lemma 1. *Let $m \in [0, N]$ be fixed. Let (x, u) be a solution of (S) and suppose that there exist $c, \alpha \in \mathbb{R}^n$ such that $x_{m+1} = \mathscr{B}_m c + \alpha$. Then the pair $(\tilde{x}, \tilde{u})$ defined by*

$$\tilde{x}_k := \begin{cases} x_k & \text{for } k \in [0, m], \\ \Phi_{k,m+1}\alpha & \text{for } k \in [m + 1, N + 1], \end{cases}$$

$$\tilde{u}_k := \begin{cases} u_k & \text{for } k \in [0, m - 1], \\ u_m - c & \text{for } k = m, \\ 0 & \text{for } k \in [m + 1, N + 1] \end{cases}$$

is admissible on $[0, N]$.

The next lemma is a symplectic analogue of [16, Lemma 2.5]. Its proof is similar to [4, Proposition 1(iii)].

Lemma 2. *Let $m \in [0, N]$ and (X, U) be the natural conjoined basis of (S). If* $\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m$ *and if $(m, m + 1]$ is not strictly conjugate to 0, then $P_m \geqslant 0$, and hence (X, U) has no focal points in $(m, m + 1]$.*

3. Coupled intervals for separated endpoints

In this section we will consider quadratic functionals of the form

$$\mathscr{F}(x, u) := x_0^{\mathrm{T}}\Gamma_0 x_0 + x_{N+1}^{\mathrm{T}}\Gamma_1 x_{N+1} + \mathscr{F}_0(x, u).$$

The functional $\mathscr{F}$ is *nonnegative* (or nonnegative definite, we write $\mathscr{F} \geqslant 0$) if $\mathscr{F}(x, u) \geqslant 0$ for all admissible pairs (x, u) satisfying $\mathscr{M}_0 x_0 = 0$, $\mathscr{M}_1 x_{N+1} = 0$. The functional $\mathscr{F}$ is *positive* (or positive definite, we write $\mathscr{F} > 0$) if $\mathscr{F}(x, u) > 0$ for all admissible pairs (x, u) satisfying $\mathscr{M}_0 x_0 = 0$, $\mathscr{M}_1 x_{N+1} = 0$, and $x \not\equiv 0$.

3.1. Coupled intervals

First we define a generalized coupled interval, in short g-coupled interval, used for the positivity of $\mathscr{F}$. Then we introduce the notion of a degenerate sequence. Finally, we define a coupled interval which is used for the nonnegativity of $\mathscr{F}$ and which is a strengthening of the g-coupled interval notion to a certain nondegenerate case.

For $m \in [0, N]$ we set

$$\mathscr{M}_1^{(m+1)} := \mathscr{M}_1 \Phi_{N+1,m+1},$$

$$\Gamma_1^{(m+1)} := \Phi_{N+1,m+1}^{\mathrm{T}} \Gamma_1 \Phi_{N+1,m+1} + \sum_{k=m+1}^{N} \Phi_{k,m+1}^{\mathrm{T}} \mathscr{C}_k^{\mathrm{T}} \Phi_{k+1,m+1}$$

(note the shift in the index k in one of the matrices Φ in the sum above). Observe that $\mathscr{M}_1^{(N+1)} = \mathscr{M}_1$ and $\Gamma_1^{(N+1)} = \Gamma_1$.

Definition 1 (*G-coupled interval, separated endpoints*). Let $m \in [0, N]$. An interval $(m, m+1]$ is *g-coupled with* 0 if there exists a solution (x, u) of (S) which satisfies (3) and there exist vectors $c, \alpha \in \mathbb{R}^n$ such that

$$\mathscr{M}_1^{(m+1)} \alpha = 0, \quad (x_m, \Phi_{N+1,m+1}\alpha) \neq 0, \quad x_{m+1} = \mathscr{B}_m c + \alpha, \quad d_m \leqslant 0,$$

where $d_m = d_m(x_m, u_{m+1}, c, \alpha)$ is defined by

$$d_m := x_m^{\mathrm{T}} c + (u_{m+1} - \mathscr{D}_m c)^{\mathrm{T}} \alpha + \alpha^{\mathrm{T}} \Gamma_1^{(m+1)} \alpha.$$

If $(m, m+1]$ is g-coupled with 0 and the corresponding solution (x, u) satisfies $d_m < 0$, then $(m, m+1]$ is *strictly g-coupled with* 0.

When $\alpha = 0$, the above (strict) g-coupled interval notion reduces to the (strict) *conjugate interval* used in [13,15], see also Section 2.

Remark 1. There is an alternative expression for d_m using the coefficients of the block tridiagonal matrix representation of the quadratic functional $\mathscr{F}$. See [13,14] for such representations. Let $\mathscr{E}_k$ be any symmetric matrix satisfying $\mathscr{D}_k^{\mathrm{T}} \mathscr{B}_k = \mathscr{B}_k^{\mathrm{T}} \mathscr{E}_k \mathscr{B}_k$, for example $\mathscr{E}_k = \mathscr{B}_k \mathscr{B}_k^{\dagger} \mathscr{D}_k \mathscr{B}_k^{\dagger}$. Let (X, U) be the natural conjoined basis of (S). Then

$$d_m = \begin{pmatrix} x_m \\ \alpha \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U_m X_m^{\dagger} + \mathscr{T}_m - \mathscr{E}_{m-1} & \mathscr{S}_m \\ \mathscr{S}_m^{\mathrm{T}} & \Gamma_1^{(m+1)} + \mathscr{E}_m \end{pmatrix} \begin{pmatrix} x_m \\ \alpha \end{pmatrix},$$

where

$$\mathscr{T}_m := \mathscr{A}_m^{\mathrm{T}} \mathscr{E}_m \mathscr{A}_m - \mathscr{A}_m^{\mathrm{T}} \mathscr{C}_m + \mathscr{E}_{m-1}, \quad \mathscr{S}_m := \mathscr{C}_m^{\mathrm{T}} - \mathscr{A}_m^{\mathrm{T}} \mathscr{E}_m$$

with $\mathscr{T}_{N+1} := \Gamma_1 + \mathscr{E}_N$ and $\mathscr{E}_{-1} := \Gamma_0$.

Definition 2 (*Degeneracy*, *separated endpoints*). Let $m \in [0, N]$. A sequence $x = \{x_k\}_{k=0}^{m+1}$ is *degenerate on* $[m+1, N+1]$ if there exists $w = \{w_k\}_{k=0}^{m}$ such that, when (uniquely) extended to $[0, N+1]$, the pair (x, w) solves (S),

$$\mathscr{M}_0 x_0 = 0, \quad w_0 = \Gamma_0 x_0 + \mathscr{M}_0 \bar{\gamma}_0, \quad \mathscr{M}_1^{(m+1)} x_{m+1} = 0$$

for some vector $\bar{\gamma}_0 \in \mathbb{R}^n$, and satisfies $\mathscr{B}_k w_k \equiv 0$ on $[m+1, N]$, i.e., $x_k = \Phi_{k,m+1} x_{m+1}$ for all $k \in [m+1, N+1]$ (void if $m = N$). Moreover, for every other $\tilde{w} = \{\tilde{w}_k\}_{k=0}^{m}$ such that $(x, \tilde{w})$ solves (S) and $\tilde{w}_0 = \Gamma_0 x_0 + \mathscr{M}_0 \tilde{\gamma}_0$ for some vector $\tilde{\gamma}_0 \in \mathbb{R}^n$ we have

$$x_m^{\mathrm{T}}(\tilde{w}_m - w_m) \geqslant 0. \tag{4}$$

If $x = \{x_k\}_{k=0}^{m+1}$ is not degenerate on $[m+1, N+1]$, then it is called *nondegenerate on* $[m+1, N+1]$.

Remark 2

(i) Since both (x, w) and $(x, \tilde{w})$ solve (S) on $[0, m]$, then $\mathscr{B}_k w_k = \mathscr{B}_k \tilde{w}_k$ on $[0, m]$.

(ii) If we denote $\Omega_{k,j} := \mathscr{D}_{k-1}\mathscr{D}_{k-2} \dots \mathscr{D}_j$ for $k > j$ and $\Omega_{k,k} := I$, then the second equation of (S) yields $\tilde{w}_m - w_m = \Omega_{m,0}(\tilde{w}_0 - w_0)$. Hence, by using the initial transversality conditions, inequality (4) takes the form

$$x_m^{\mathrm{T}} \Omega_{m,0} \mathscr{M}_0 (\tilde{\gamma}_0 - \bar{\gamma}_0) \geqslant 0.$$

(iii) Inequality (4) can be eliminated in the definition of a degenerate sequence $x = \{x_k\}_{k=0}^{m+1}$ if one of the following conditions holds.

(a) System (S) is $(\mathscr{M}_0 : I)$-normal on $[0, N+1]$. In this case $\tilde{w}_k$ and w_k must coincide.

(b) $m = 0$. We have in this case

$$x_0^{\mathrm{T}}(\tilde{w}_0 - w_0) = (\mathscr{M}_0 x_0)^{\mathrm{T}} (\tilde{\gamma}_0 - \bar{\gamma}_0) = 0.$$

(c) $\mathscr{M}_0 = 0$, i.e., the left endpoint is free. This follows from part (ii) of this remark.

(d) $\mathscr{B}_k$ is invertible for some $k \in [0, m]$. Under this condition, part (i) of this remark implies that $w_k = \tilde{w}_k$ at this index k. Hence, the uniqueness of the solutions of (S) yields $\tilde{w}_j = w_j$ for all $j \in [0, m]$.

Definition 3 (*Coupled interval*, *separated endpoints*). Let $m \in [0, N]$. An interval $(m, m+1]$ is called *coupled with* 0 if it is g-coupled with 0 for some (x, u), c, and α, and if the sequence $\tilde{x} := (\{x_k\}_{k=0}^{m}, \alpha)$ is nondegenerate on $[m+1, N+1]$ (drop if $m = N$). If $(m, m+1]$ is coupled with 0 and the corresponding solution (x, u) satisfies $d_m < 0$, then $(m, m+1]$ is called *strictly coupled with* 0.

Remark 3. If the system (S) is a Hamiltonian system (H), that is, if the coefficients of S_k are

$$\begin{aligned} &\mathscr{A}_k = \tilde{A}_k := (I - A_k)^{-1}, \quad \mathscr{B}_k = \tilde{A}_k B_k, \quad \mathscr{C}_k = C_k \tilde{A}_k, \\ &\mathscr{D}_k = C_k \tilde{A}_k B_k + I - A_k^{\mathrm{T}}, \end{aligned} \tag{5}$$

then the above (strictly) g-coupled and coupled interval notions reduce to the corresponding notions from [16].

Example 1. The introduction of coupled intervals for discrete symplectic system (S) allows us to consider this notion for *discrete trigonometric* or *self-reciprocal systems*, see e.g. [2,6]. An example of such a system is when we take $S_k \equiv \mathscr{J}$, i.e., $\mathscr{A}_k = \mathscr{D}_k \equiv 0$ and $\mathscr{B}_k = -\mathscr{C}_k \equiv I$. Note that in this case $\Phi_{k,j} = 0$ for $k > j$ (and still $\Phi_{k,k} = I$), $\mathscr{M}_1^{(m+1)} = 0 = \Gamma_1^{(m+1)}$ for $m \in [0, N-1]$ while $\mathscr{M}_1^{(N+1)} = \mathscr{M}_1$ and $\Gamma_1^{(N+1)} = \Gamma_1$, the vector $c \in \mathbb{R}^n$ in Definition 3 is uniquely determined by the solution (x, u) of (S) and $\alpha \in \mathbb{R}^n$, and $d_m = x_m^{\mathrm{T}} x_{m+1} - 2x_m^{\mathrm{T}} \alpha$. Coupled intervals for this type of system could not be defined via the Hamiltonian formulation of coupled intervals in [16], due to the nonexistence of coefficients A_k, B_k, and C_k satisfying equations (5) in this setting.

In the remaining part of this section we will present results which relate the (strict) g-coupled and coupled intervals to the definiteness of the quadratic functional $\mathscr{F}$.

Proposition 1. *Let* (X, U) *be the natural conjoined basis of* (S).

(i) *Suppose that* $(N, N+1]$ *is not coupled with* 0. *Then, for* $m \in [0, N]$, *if* $(m, m+1]$ *is g-coupled with* 0 *and if either* $\Phi_{N+1,m+1}\alpha \neq 0$, *or* $\Phi_{N+1,m+1}\alpha = 0$ *and* $\operatorname{Ker} X_{N+1} \subseteq \operatorname{Ker} X_m$, *then* $(m, m+1]$ *is also coupled with* 0.

(ii) *Suppose that* $(N, N+1]$ *is not strictly coupled with* 0. *Then, for* $m \in [0, N]$, *if* $(m, m+1]$ *is strictly g-coupled with* 0, *then* $(m, m+1]$ *is also strictly coupled with* 0.

Proof. If $m = N$ the degeneracy is not needed in Definition 3, and hence the notion of a (strict) coupled interval coincides with that of a (strict) g-coupled interval. Thus, in this case, the statements in both conditions (i) and (ii) above hold true. Therefore, take $m \in [0, N-1]$ in the rest of the proof.

(i) Let $(m, m+1]$ be g-coupled with 0 with the corresponding (x, u), c, α, and $d_m \leqslant 0$, and assume that $\tilde{x} := (\{x_k\}_{k=0}^m, \alpha)$ is degenerate on $[m+1, N+1]$ with the corresponding $w = \{w_k\}_{k=0}^m$. Then $(\tilde{x}, w)$ solves (S), $\mathscr{B}_k w_k \equiv 0$ on $[m+1, N]$, $\tilde{x}$ has the form as in Lemma 1, and inequality (4) with $\tilde{w} := u$ yields $\tilde{x}_m^{\mathrm{T}}(u_m - w_m) \geqslant 0$. We will show that this leads to a contradiction with $(N, N+1]$ not coupled with 0. Equations

$$x_{m+1} = \mathscr{B}_m c + \alpha = \mathscr{A}_m \tilde{x}_m + \mathscr{B}_m u_m, \quad \alpha = \tilde{x}_{m+1} = \mathscr{A}_m \tilde{x}_m + \mathscr{B}_m w_m \tag{6}$$

yield

$$\mathscr{B}_m (u_m - c) = \mathscr{B}_m w_m. \tag{7}$$

Now use that $\mathscr{B}_k w_k \equiv 0$ on $[m+1, N]$ in the system (S) for $k \in [m, N]$, the first identity in (1), and add the system equations to get

$$\tilde{x}_{N+1}^{\mathrm{T}} w_{N+1} - \tilde{x}_m^{\mathrm{T}} w_m = 2\tilde{x}_m^{\mathrm{T}} \mathscr{C}_m^{\mathrm{T}} \mathscr{B}_m w_m + w_m^{\mathrm{T}} \mathscr{B}_m^{\mathrm{T}} \mathscr{D}_m w_m + \sum_{k=m}^{N} \tilde{x}_k^{\mathrm{T}} \mathscr{A}_k^{\mathrm{T}} \mathscr{C}_k \tilde{x}_k. \tag{8}$$

From Eqs. (6)–(8) and system (S) it follows that d_m from Definition 3 reduces to

$$\begin{aligned} d_m &= \tilde{x}_m^{\mathrm{T}} u_m - \tilde{x}_m^{\mathrm{T}}(u_m - c) + (\mathscr{C}_m \tilde{x}_m + \mathscr{D}_m (u_m - c))^{\mathrm{T}} (\mathscr{A}_m \tilde{x}_m + \mathscr{B}_m w_m) \\ &\quad + \alpha^{\mathrm{T}} \Gamma_1^{(m+1)} \alpha \\ &= \tilde{x}_m^{\mathrm{T}} u_m + (u_m - c)^{\mathrm{T}} (\mathscr{D}_m^{\mathrm{T}} \mathscr{A}_m - I) \tilde{x}_m + \tilde{x}_m^{\mathrm{T}} \mathscr{C}_m^{\mathrm{T}} (\mathscr{A}_m \tilde{x}_m + \mathscr{B}_m w_m) \\ &\quad + w_m^{\mathrm{T}} \mathscr{B}_m^{\mathrm{T}} \mathscr{D}_m w_m + \tilde{x}_{N+1}^{\mathrm{T}} \Gamma_1 \tilde{x}_{N+1} + \sum_{k=m+1}^{N} \tilde{x}_k^{\mathrm{T}} \mathscr{C}_k^{\mathrm{T}} \mathscr{A}_k \tilde{x}_k \\ &= \tilde{x}_m^{\mathrm{T}} u_m + 2\tilde{x}_m^{\mathrm{T}} \mathscr{C}_m^{\mathrm{T}} \mathscr{B}_m w_m + w_m^{\mathrm{T}} \mathscr{B}_m^{\mathrm{T}} \mathscr{D}_m w_m + \tilde{x}_{N+1}^{\mathrm{T}} \Gamma_1 \tilde{x}_{N+1} + \sum_{k=m}^{N} \tilde{x}_k^{\mathrm{T}} \mathscr{A}_k^{\mathrm{T}} \mathscr{C}_k \tilde{x}_k \\ &= \tilde{x}_m^{\mathrm{T}} (u_m - w_m) + \tilde{x}_{N+1}^{\mathrm{T}} \Gamma_1 \tilde{x}_{N+1} + \tilde{x}_{N+1}^{\mathrm{T}} w_{N+1} \end{aligned}$$

$$= \tilde{x}_m^{\mathrm{T}}(u_m - w_m) + \tilde{x}_{N+1}^{\mathrm{T}}\tilde{c} + (w_{N+1} - \mathscr{D}_N\tilde{c})^{\mathrm{T}}\tilde{\alpha} + \tilde{\alpha}^{\mathrm{T}}\Gamma_1^{(N+1)}\tilde{\alpha}$$
$$= \tilde{x}_m^{\mathrm{T}}(u_m - w_m) + \tilde{d}_N(\tilde{x}_N, w_{N+1}, \tilde{c}, \tilde{\alpha}),$$

where we set $\tilde{c} := 0$ and $\tilde{\alpha} := \tilde{x}_{N+1}$. Hence, $\mathscr{M}_1^{(N+1)}\tilde{\alpha} = \mathscr{M}_1^{(m+1)}\alpha = 0$, $\tilde{x}_{N+1} = \mathscr{B}_N\tilde{c} + \tilde{\alpha}$, and $\tilde{d}_N \leqslant d_m \leqslant 0$.

If we now assume that $\Phi_{N+1,m+1}\alpha \neq 0$, then $\tilde{\alpha} = \tilde{x}_{N+1} \neq 0$ so that $(\tilde{x}_N, \Phi_{N+1,N+1}\tilde{\alpha}) \neq 0$. Thus, in this case $(N, N+1]$ is coupled with 0.

Finally, suppose $\tilde{x}_{N+1} = \Phi_{N+1,m+1}\alpha = 0$ and $\operatorname{Ker} X_{N+1} \subseteq \operatorname{Ker} X_m$. Since $(\tilde{x}, w)$ solves (S) and $\mathscr{M}_0\tilde{x}_0 = 0$, $w_0 = \Gamma_0\tilde{x}_0 + \mathscr{M}_0\bar{\gamma}_0$, it follows that $(\tilde{x}_k, w_k) = (X_k\beta, U_k\beta)$ for all $k \in [0, N+1]$ (with $\beta := x_0 + \mathscr{M}_0\bar{\gamma}_0$), because both are solutions of (S) satisfying the same initial conditions. Hence, $X_{N+1}\beta = \tilde{x}_{N+1} = 0$ and from $\operatorname{Ker} X_{N+1} \subseteq \operatorname{Ker} X_m$ we get $\tilde{x}_m = X_m\beta = 0$. Thus $(x_m, \Phi_{N+1,m+1}\alpha) = (\tilde{x}_m, \tilde{x}_{N+1}) = 0$. This contradicts the fact that $(m, m+1]$ is g-coupled with 0.

(ii) Let $(m, m+1]$ be strictly g-coupled with 0 with the corresponding (x, u), c, α, and $d_m < 0$. Then in the same way as in part (i) we obtain $\tilde{d}_N \leqslant d_m < 0$, which means that $(N, N+1]$ is strictly coupled with 0. This is a contradiction. □

The following result is a direct consequence of the above proposition.

Corollary 1. *Let (X, U) be the natural conjoined basis of* (S).

(i) *Assume that $(N, N+1]$ is not coupled with 0 and $\operatorname{Ker} X_{N+1} \subseteq \operatorname{Ker} X_k$ holds for all $k \in [0, N]$. Then there is no interval coupled with 0 in $(0, N+1]$ if and only if there is no interval g-coupled with 0 in $(0, N+1]$.*

(ii) *Assume that $(N, N+1]$ is not strictly coupled with 0. Then there is no interval strictly coupled with 0 in $(0, N+1]$ if and only if there is no interval strictly g-coupled with 0 in $(0, N+1]$.*

3.2. Nonnegativity via coupled intervals

Next we present necessary conditions for $\mathscr{F} \geqslant 0$ and several characterizations of $\mathscr{F} \geqslant 0$.

Theorem 1 (Necessary conditions for $\mathscr{F} \geqslant 0$). *Suppose that $\mathscr{F} \geqslant 0$ over $\mathscr{M}_0x_0 = 0$ and $\mathscr{M}_1x_{N+1} = 0$. Then the following conditions hold under the corresponding assumptions.*

(i) *$(0, N+1]$ contains no intervals strictly g-coupled with 0 or, equivalently, no intervals strictly coupled with 0.*

(ii) *If the system (S) is $(\mathscr{M}_0 : I)$-normal on $[0, N+1]$, then $(0, N]$ contains no intervals coupled with 0, and $(N, N+1]$ is not strictly coupled with 0.*

Proof. (i) If there exists $m \in [0, N]$ such that $(m, m+1]$ is strictly g-coupled with 0, then with the corresponding solution (x, u) of (S) and with the admissible $(\tilde{x}, \tilde{u})$ from Lemma 1 we have $\mathscr{F}(\tilde{x}, \tilde{u}) = d_m < 0$. This is a contradiction. By Corollary 1(ii) we get the equivalence with the nonexistence of strictly coupled intervals in $(0, N+1]$.

(ii) This proof goes along the same line as the proof [16, Theorem 3.15]. □

Theorem 2 (Characterization of $\mathscr{F} \geqslant 0$). *Assume that* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *holds for all* $k \in [0, N-1]$. *Then the following are equivalent.*

(i) $\mathscr{F} \geqslant 0$ *over* $\mathscr{M}_0\eta_0 = 0$ *and* $\mathscr{M}_1\eta_{N+1} = 0$.
(ii) $(0, N+1]$ *contains no intervals strictly coupled with* 0.
(iii) $(0, N]$ *contains no intervals strictly conjugate to* 0, *and* $(N, N+1]$ *is not strictly coupled with* 0.

Proof. (i) ⇒ (ii) This is Theorem 1(i).

(ii) ⇒ (iii) Since $(N, N+1]$ is not strictly coupled with 0, we get from Corollary 1(ii) that $(0, N]$ contains no intervals strictly g-coupled with 0. This yields the nonexistence of strictly conjugate intervals to 0 in $(0, N]$, in fact in $(0, N+1]$.

(iii) ⇒ (i) By Lemma 2, we have that (X, U) has no focal points in $(0, N]$. Then $\mathscr{F} \geqslant 0$ follows via [13, Theorem 3(ii)] with $m = N$. □

If, in addition to the kernel condition on $[0, N-1]$, the system (S) is $(\mathscr{M}_0 : I)$-normal, then the *strictly coupled* interval condition in $(0, N]$ in Theorem 2 can be replaced by the *coupled* interval condition therein.

Corollary 2 (Characterization of $\mathscr{F} \geqslant 0$). *Let* (X, U) *be the natural conjoined basis of* (S). *Assume that* (S) *is* $(\mathscr{M}_0 : I)$*-normal on* $[0, N+1]$ *and* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *holds for all* $k \in [0, N-1]$. *Then each condition* (i)–(iii) *in Theorem 2 is also equivalent to*

(iv) $(0, N]$ *contains no intervals coupled with* 0, *and* $(N, N+1]$ *is not strictly coupled with* 0.

3.3. Positivity via coupled intervals

The following theorem characterizes the positivity of $\mathscr{F}$ in terms of the nonexistence of g-coupled intervals in $(0, N+1]$.

Theorem 3 (Characterization of $\mathscr{F} > 0$). *The following conditions are equivalent*:

(i) $\mathscr{F} > 0$ *over* $\mathscr{M}_0x_0 = 0$, $\mathscr{M}_1x_{N+1} = 0$, *and* $x \not\equiv 0$.
(ii) *There is no interval g-coupled with* 0 *in* $(0, N+1]$.
(iii) *There is no interval conjugate to* 0 *in* $(0, N]$, *and* $(N, N+1]$ *is not coupled with* 0.
(iv) *There is no interval coupled with* 0 *in* $(0, N+1]$, *and* $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ *for all* $k \in [0, N]$, *where* (X, U) *is the natural conjoined basis of* (S).

Proof. (i) ⇒ (ii) If $(m, m+1]$ is g-coupled with 0 for some $m \in [0, N]$, then with the corresponding solution (x, u) of (S) and with the admissible $(\tilde{x}, \tilde{u})$ from Lemma 1 we have $\mathscr{F}(\tilde{x}, \tilde{u}) = d_m \leqslant 0$. Moreover, since $(\tilde{x}_m, \tilde{x}_{N+1}) = (x_m, \Phi_{N+1,m+1}\alpha) \neq 0$, we have $\tilde{x} \not\equiv 0$. This contradicts $\mathscr{F} > 0$.

(ii) ⇒ (iii) This is trivial, since every conjugate interval to 0 is g-coupled with 0, and $(N, N+1]$ not coupled with 0 is the same as $(N, N+1]$ not g-coupled with 0.

(iii) ⇒ (i) This implication follows via [15, Theorem 5(ii)].

(i) $\Rightarrow$ (iv) The nonexistence of coupled intervals in $(0, N+1]$ follows from part (ii), and $\mathscr{F} > 0$ implies the kernel condition on $[0, N]$ e.g. by [15, Theorem 5(iii)].

(iv) $\Rightarrow$ (ii) This follows from Corollary 1(i). $\square$

4. Coupled intervals for joint endpoints

In this section we will study the coupled interval notions for problems with general boundary conditions, namely for the quadratic functional

$$\mathscr{F}(x,u) := \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^{\mathrm{T}} \Gamma \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} + \mathscr{F}_0(x,u)$$

with jointly varying endpoints $\mathscr{M} \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} = 0$. It is well known that this quadratic functional can be transformed (in two ways) into a quadratic functional in double dimension with separated endpoints, see e.g. [15,16]. These two augmentations result into two coupled interval notions for joint endpoints which are both displayed below. Applying the results on separated endpoints from the previous section to these augmented problems we obtain the corresponding statements pertaining $\mathscr{F} \geqslant 0$ and $\mathscr{F} > 0$ with joint endpoints in terms of such coupled intervals. However, these results are not repeated below in full details.

For $m \in [0, N]$ we set

$$\begin{aligned} \mathscr{M}^{(m+1)} &:= \mathscr{M} \begin{pmatrix} I & 0 \\ 0 & \Phi_{N+1,m+1} \end{pmatrix}, \\ \Gamma^{(m+1)} &:= \begin{pmatrix} I & 0 \\ 0 & \Phi_{N+1,m+1}^{\mathrm{T}} \end{pmatrix} \Gamma \begin{pmatrix} I & 0 \\ 0 & \Phi_{N+1,m+1} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & \sum_{k=m+1}^{N} \Phi_{k,m+1}^{\mathrm{T}} \mathscr{C}_k^{\mathrm{T}} \Phi_{k+1,m+1} \end{pmatrix}. \end{aligned}$$

4.1. Coupled intervals

The following definitions come from moving the augmented boundary conditions to 0, see the $\mathscr{F}^{\#}$ augmentation in [15,16].

Definition 4 (*G-coupled interval, joint endpoints*). Let $m \in J$. An interval $(m, m+1]$ is called *g-coupled with* 0 if there exist a solution (x, u) of (S) and vectors $c, \alpha, \beta \in \mathbb{R}^n$ such that

$$\mathscr{M}^{(m+1)} \begin{pmatrix} x_0 \\ \alpha \end{pmatrix} = 0, \quad \begin{pmatrix} u_0 \\ \beta \end{pmatrix} = \Gamma \begin{pmatrix} x_0 \\ \Phi_{N+1,m+1}\alpha \end{pmatrix} + \mathscr{M}\gamma$$

for some vector $\gamma \in \mathbb{R}^{2n}$, and satisfying

$$(x_m, \Phi_{N+1,m+1}\alpha) \neq 0, \quad x_{m+1} = \mathscr{B}_m c + \alpha, \quad d_m \leqslant 0,$$

where $d_m = d_m(x_m, u_{m+1}, c, \alpha, \beta)$ is defined by

$$d_m := x_m^{\mathrm{T}} c + (u_{m+1} - \mathscr{D}_m c)^{\mathrm{T}} \alpha + \beta^{\mathrm{T}} \Phi_{N+1,m+1}\alpha + \alpha^{\mathrm{T}} \sum_{k=m+1}^{N} \Phi_{k,m+1}^{\mathrm{T}} \mathscr{C}_k^{\mathrm{T}} \Phi_{k+1,m+1}\alpha.$$

If $(m, m+1]$ is g-coupled with 0 and $d_m < 0$, then $(m, m+1]$ is called *strictly g-coupled with* 0.

Definition 5 (*Degeneracy*, *joint endpoints*). Let $m \in J$. A sequence $x = \{x_k\}_{k=0}^{m+1}$ is said to be *degenerate on* $[m+1, N+1]$ if there exist $w = \{w_k\}_{k=0}^{m}$ and $\rho \in \mathbb{R}^n$ such that (x, w) solves (S),

$$\mathscr{M}^{(m+1)} \begin{pmatrix} x_0 \\ x_{m+1} \end{pmatrix} = 0, \quad \begin{pmatrix} w_0 \\ \rho \end{pmatrix} = \Gamma \begin{pmatrix} x_0 \\ \Phi_{N+1,m+1} x_{m+1} \end{pmatrix} + \mathscr{M} \bar{\gamma}$$

for some vector $\bar{\gamma} \in \mathbb{R}^{2n}$, and satisfies $\mathscr{B}_k w_k \equiv 0$ on $[m+1, N]$ (void if $m = N$). Moreover, for every other $\tilde{w} = \{\tilde{w}_k\}_{k=0}^{m}$ and $\tilde{\rho} \in \mathbb{R}^n$ such that $(\eta, \tilde{w})$ solves (S) and

$$\begin{pmatrix} \tilde{w}_0 \\ \tilde{\rho} \end{pmatrix} = \Gamma \begin{pmatrix} x_0 \\ \Phi_{N+1,m+1} x_{m+1} \end{pmatrix} + \mathscr{M} \tilde{\gamma},$$

for some vector $\tilde{\gamma} \in \mathbb{R}^{2n}$, we have

$$x_m^{\mathrm{T}} (\tilde{w}_m - w_m) - x_0^{\mathrm{T}} (\tilde{w}_0 - w_0) \geqslant 0. \tag{9}$$

If $x = \{x_k\}_{k=0}^{m+1}$ is not degenerate on $[m+1, N+1]$, then it is called *nondegenerate on* $[m+1, N+1]$.

Definition 6 (*Coupled interval*, *joint endpoints*). Let $m \in J$. An interval $(m, m+1]$ is called *coupled with* 0 if it is g-coupled with 0 (according to Definition 4) with the corresponding (x, u), c, α, and β, and if the sequence $\tilde{x} := (\{x_k\}_{k=0}^{m}, \alpha)$ is nondegenerate on $[m+1, N+1]$ (according to Definition 5, drop if $m = N$). If $(m, m+1]$ is coupled with 0 and $d_m < 0$, then it is called *strictly coupled with* 0.

When the endpoints of $\mathscr{F}$ are separable, i.e., when $\mathscr{M} = \operatorname{diag}\{\mathscr{M}_0, \mathscr{M}_1\}$ and $\Gamma = \operatorname{diag}\{\Gamma_0, \Gamma_1\}$, then these definitions reduce to the corresponding ones in Definitions 1–3. Hence, they are direct extensions to jointly varying endpoints of the (strictly) coupled interval notions for separable endpoints.

Now the statements in the previous section hold for $\mathscr{F} \geqslant 0$ and $\mathscr{F} > 0$ over $\mathscr{M} \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} = 0$ with the following modifications:

- $(\mathscr{M}_0 : I)$-normality of (S) in Theorem 1 and Corollary 2 is replaced by the condition that the only solution of $u_{k+1} = \mathscr{D}_k u_k$, $\mathscr{B}_k u_k = 0$, $k \in [0, N]$, satisfying $\begin{pmatrix} u_0 \\ \beta \end{pmatrix} = \mathscr{M} \gamma$, for some vectors $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^{2n}$, is $u_k \equiv 0$ on $[0, N+1]$ and $\beta = 0$.
- The natural conjoined basis (X, U) of (S) in Theorems 2, 3 and Corollary 2 is replaced by the augmented natural conjoined basis $(X^{\#}, U^{\#})$ which starts with the initial conditions $X_0^{\#} = I - \mathscr{M}$ and $U_0^{\#} = \Gamma + \mathscr{M}$.

4.2. Coupled* intervals

The following definitions come from moving the augmented boundary conditions to $N+1$, see the $\mathscr{F}^*$ augmentation in [15,16].

Definition 7 (*G-coupled* interval*, *joint endpoints*). Let $m \in J$. An interval $(m, m+1]$ is *g-coupled* with* 0 if there exist a solution (x, u) of (S) and vectors $c, \alpha \in \mathbb{R}^n$ such that

$$\mathscr{M}^{(m+1)}\begin{pmatrix} x_0 \\ \alpha \end{pmatrix} = 0, \quad (x_0, x_m, \Phi_{N+1,m+1}\alpha) \neq 0, \quad x_{m+1} = \mathscr{B}_m c + \alpha, \quad d_m^* \leqslant 0,$$

where $d_m^* = d_m^*(x_m, u_{m+1}, c, \alpha, x_0, u_0)$ is defined by

$$d_m^* := x_m^{\mathrm{T}} c + (u_{m+1} - \mathscr{D}_m c)^{\mathrm{T}} \alpha + \begin{pmatrix} x_0 \\ \alpha \end{pmatrix}^{\mathrm{T}} \Gamma^{(m+1)} \begin{pmatrix} x_0 \\ \alpha \end{pmatrix} - x_0^{\mathrm{T}} u_0.$$

If $(m, m+1]$ is g-coupled* with 0 and $d_m^* < 0$, then it is called *strictly g-coupled* with* 0.

Definition 8 (*Degeneracy**, *joint endpoints*). Let $m \in J$. A sequence $x = \{x_k\}_{k=0}^{m+1}$ is *degenerate* on* $[m+1, N+1]$ if there exists $w = \{w_k\}_{k=0}^{m}$ such that (x, w) solves (S),

$$\mathscr{M}^{(m+1)}\begin{pmatrix} x_0 \\ x_{m+1} \end{pmatrix} = 0, \quad \mathscr{B}_k w_k \equiv 0 \quad \text{on } [m+1, N]$$

and for every other $\tilde{w} = \{\tilde{w}_k\}_{k=0}^{m}$ such that $(x, \tilde{w})$ solves (S) inequality (9) holds. If $x = \{x_k\}_{k=0}^{m+1}$ is not degenerate* on $[m+1, N+1]$, then it is called *nondegenerate* on* $[m+1, N+1]$.

Definition 9 (*Coupled* interval*, *joint endpoints*). Let $m \in J$. An interval $(m, m+1]$ is called *coupled* with* 0 if it is g-coupled* with 0 (according to Definition 7) with some solution (x, u) of (S) and vectors $c, \alpha \in \mathbb{R}^n$, if the sequence $\tilde{x} := (\{x_k\}_{k=0}^{m}, \alpha)$ is nondegenerate* on $[m+1, N+1]$ (according to Definition 8, drop if $m = N$). If $(m, m+1]$ is coupled* with 0 and $d_m^* < 0$, then it is called *strictly coupled* with* 0.

The statements in the previous section hold for $\mathscr{F} \geqslant 0$ and $\mathscr{F} > 0$ over $\mathscr{M}\begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} = 0$ with the following modifications:

- strict (g-)coupled intervals are replaced by the corresponding (strict) (g-)coupled* intervals,
- $(\mathscr{M}_0 : I)$-normality of (S) in Theorem 1 and Corollary 2 is replaced by the $(I : I)$-normality of (S),
- the natural conjoined basis (X, U) of (S) in Theorems 2, 3 and Corollary 2 is replaced by the principal solution $(\widehat{X}, \widehat{U})$ of (S).

It is interesting to note that the above (strict) g-coupled* interval generates naturally new (strict) conjugate* interval notion upon taking $\alpha = 0$ (and hence $\mathscr{M} = \operatorname{diag}\{\mathscr{M}_0, \star\}$, $\Gamma = \operatorname{diag}\{\Gamma_0, \star\}$), compare with the (strict) conjugate interval notion in Section 2, as follows. An interval $(m, m+1]$ is (*strictly*) *conjugate* to* 0 if there exist a solution (x, u) of (S) and vector $c \in \mathbb{R}^n$ such that

$$\mathscr{M}_0 x_0 = 0, \quad (x_0, x_m) \neq 0, \quad x_{m+1} = \mathscr{B}_m c, \quad d_m^* \leqslant 0 \ (d_m^* < 0),$$

where $d_m^* = d_m^*(x_m, c, x_0, u_0) := x_m^{\mathrm{T}} c + x_0^{\mathrm{T}}(\Gamma_0 x_0 - u_0)$. Thus we obtain another characterization of the positivity of $\mathscr{F}$ with fixed right endpoint or separated endpoints in terms of this conjugate* interval notion.

Corollary 3 (Characterization of $\mathscr{F} > 0$, fixed right endpoint). *The functional* $\mathscr{F} > 0$ *over* $\mathscr{M}_0 x_0 = 0$, $x_{N+1} = 0$, *and* $x \not\equiv 0$ *if and only if there is no interval conjugate* to* 0 *in* $(0, N+1]$.

Corollary 4 (Characterization of $\mathscr{F} > 0$, separated endpoints). *The functional* $\mathscr{F} > 0$ *over* $\mathscr{M}_0 x_0 = 0$, $\mathscr{M}_1 x_{N+1} = 0$, *and* $x \not\equiv 0$ *if and only if there is no interval conjugate* to* 0 *in* $(0, N+1]$, *and* $(N, N+1]$ *is not coupled* with* 0.

Remark 4. Let us point out that the above (strict) (g-)coupled and (g-)coupled* interval notions presented in this section *coincide* in the case of *periodic endpoints* when $x_0 = x_{N+1}$, i.e., when $\mathscr{M} := \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ and $\Gamma := \operatorname{diag}\{\Gamma_0, 0\}$, and/or in the case of discrete *calculus of variations* [17] when the matrices $\mathscr{B}_k$ are invertible.

5. Example

Consider the following nonlinear discrete control problem of minimizing the functional

$$\mathscr{J}_N(\xi, w) := -\frac{1}{4}\xi_{N+1}^2 + \frac{1}{2}\sum_{k=0}^{N}(\xi_{k+1}^3 + w_k^2)h$$

subject to sequences $\xi = \{\xi_k\}_{k=0}^{N+1}$ and $w = \{w_k\}_{k=0}^{N}$ satisfying

$$\xi_{k+1} = \xi_k + h(1 - kh)^{1/2} w_k, \quad k \in [0, N], \ \xi_0 = 0,$$

where $h := 1/(N+1)$ and $N \geqslant 2$. By [15, Theorem 3] and [15, Proposition 1], the positivity of the second variation $\mathscr{J}_N''$ of $\mathscr{J}_N$ is a sufficient condition for the optimality of the pair $(\hat{\xi}_k, \hat{w}_k) \equiv (0, 0)$ (with the adjoint variable $\hat{p}_k \equiv 0$), where

$$\mathscr{J}_N''(x, u) := -\frac{1}{2}x_{N+1}^2 + \sum_{k=0}^{N} h(1 - kh)u_k^2$$

subject to sequences $x = \{x_k\}_{k=0}^{N+1}$ and $u = \{u_k\}_{k=0}^{N}$ satisfying

$$x_{k+1} = x_k + h(1 - kh)u_k, \quad k \in [0, N], \ x_0 = 0.$$

Note that (ξ, w) and (x, u) above are denoted in [15] by (x, u) and (η, q), respectively. We will show that $(0, N+1]$ contains no g-coupled intervals with 0, so that condition (ii) of Theorem 3 will then imply the optimality of the above given pair $(\hat{\xi}_k, \hat{w}_k)$.

We have $n = 1$, $\mathscr{A}_k = \mathscr{D}_k \equiv 1$, $\mathscr{B}_k = h(1 - kh)$, $\mathscr{C}_k \equiv 0$, $\Gamma_0 = 0$, $\Gamma_1 = -\frac{1}{2}$, $\mathscr{M}_0 = I$, $\mathscr{M}_1 = 0$. Since the solutions of (S) satisfying $x_0 = 0$ have the form

$$x_k = (1/2)h(k+1)(2 - kh)u_0, \quad k \in [1, N+1],$$
$$u_k \equiv u_0, \quad k \in [0, N+1],$$

it follows from Remark 1 that $d_0 = (-\frac{1}{2} + \frac{1}{h})\alpha^2$ and for $m \in [1, N]$

$$d_m = \begin{pmatrix} x_m \\ \alpha \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \dfrac{2}{hm[2 - (m-1)h]} + \dfrac{1}{h(1 - mh)} & -\dfrac{1}{h(1 - mh)} \\ -\dfrac{1}{h(1 - mh)} & -\dfrac{1}{2} + \dfrac{1}{h(1 - mh)} \end{pmatrix} \begin{pmatrix} x_m \\ \alpha \end{pmatrix}. \tag{10}$$

By Definition 1, the nonexistence of g-coupled intervals in $(0, N+1]$ means that $d_m > 0$ for any solution (x, u) of (S), $\alpha \in \mathbb{R}^n$, and $m \in [0, N]$ such that $x_0 = 0$ and $(x_m, \alpha) \neq 0$. Thus, for $m = 0$ we have $\alpha \neq 0$ and $d_0 = (N + \frac{1}{2})\alpha^2 > 0$. Let $m \in [1, N]$ and denote the symmetric 2×2 matrix in (10) by D. Since for $m \in [1, N]$ we require that $(x_m, \alpha) \neq 0$, it is enough to show that D is a positive definite matrix and then $d_m > 0$ follows. Condition $D > 0$ is however equivalent to $\det D > 0$, since the left upper corner of D is positive. If we denote the common denominator of $\det D$ by $q_{m,h}$, i.e., $q_{m,h} = 2h^2 m[2 - (m-1)h](1 - mh) > 0$, then we have

$$\begin{aligned}(\det D)q_{m,h} &= 4 - h(2 - mh + 2m - m^2 h) > 4 - 2h(m+1) \\ &= 4 - 2\frac{m+1}{N+1} \geqslant 2 > 0\end{aligned}$$

for all $m \in [1, N]$. Therefore, interval $(0, N+1]$ contains no g-coupled intervals with 0 and, by Theorem 3(ii), the quadratic functional J''_N is positive definite.

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References

[1] C.D. Ahlbrandt, A.C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, Kluwer Academic Publishers, Boston, 1996.
[2] D.R. Anderson, Discrete trigonometric matrix functions, PanAmer. Math. J. 7 (1) (1997) 39–54.
[3] D.R. Anderson, Normalized prepared bases for discrete symplectic matrix systems, R.P. Agarwal, M. Bohner (Eds.), Discrete and Continuous Hamiltonian Systems (special issue), Dynam. Systems Appl. 8 (3–4) (1999) 335–344.
[4] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199 (3) (1996) 804–826.
[5] M. Bohner, O. Došlý, Disconjugacy and transformations for symplectic systems, Rocky Mountain J. Math. 27 (3) (1997) 707–743.
[6] M. Bohner, O. Došlý, Trigonometric transformation of symplectic difference systems, J. Differential Equations 163 (1) (2000) 113–129.
[7] M. Bohner, O. Došlý, The discrete Prüfer transformation, Proc. Amer. Math. Soc. 129 (9) (2001) 2715–2726.
[8] M. Bohner, O. Došlý, R. Hilscher, W. Kratz, Diagonalization approach to discrete quadratic functionals, Arch. Inequal. Appl. 1 (2) (2003) 261–274.
[9] M. Bohner, O. Došlý, W. Kratz, Positive semidefiniteness of discrete quadratic functionals, Proc. Edinburgh Math. Soc. 46 (2003) 627–636.
[10] M. Bohner, O. Došlý, W. Kratz, An oscillation theorem for discrete eigenvalue problems, Rocky Mountain J. Math. 33 (4) (2003) 1233–1260.
[11] M. Bohner, A. Peterson, Dynamic Equations on Time Scales. An Introduction with Applications, Birkhäuser, Boston, 2001.
[12] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[13] O. Došlý, R. Hilscher, V. Zeidan, Nonnegativity of discrete quadratic functionals corresponding to symplectic difference systems, Linear Algebra Appl. 375 (2003) 21–44.
[14] R. Hilscher, Disconjugacy of symplectic systems and positivity of block tridiagonal matrices, Rocky Mountain J. Math. 29 (4) (1999) 1301–1319.
[15] R. Hilscher, V. Zeidan, Symplectic difference systems: variable stepsize discretization and discrete quadratic functionals, Linear Algebra Appl. 367 (2003) 67–104.

[16] R. Hilscher, V. Zeidan, Coupled intervals in the discrete optimal control, J. Difference Equ. Appl. 10 (2) (2004) 151–186.
[17] R. Hilscher, V. Zeidan, Nonnegativity and positivity of a quadratic functional in the discrete calculus of variations: A survey, J. Difference Equ. Appl. 11 (9) (2005) 857–875.
[18] W. Kratz, Discrete oscillation, J. Difference Equ. Appl. 9 (1) (2003) 135–147.
[19] V. Zeidan, P. Zezza, Coupled points in the calculus of variations and applications to periodic problems, Trans. Amer. Math. Soc. 315 (1) (1989) 323–335.
[20] V. Zeidan, P. Zezza, Coupled points in optimal control theory, IEEE Trans. Automat. Control 36 (11) (1991) 1276–1281.